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Positive and dead core solutions of singular Dirichlet boundary value problems with ϕ-Laplacian

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Abstract

The paper discusses the existence of positive solutions, dead core solutions and pseudodead core solutions of the singular Dirichlet boundary value problem $(\phi(u'))' = \lambda[f(t,u,u') + h(t,u,u')]$, $u(0) = u(T) = A$. Here λ is the positive parameter, $A > 0$, f is singular at the value 0 of its first phase variable and h may be singular at the value 0 of its second phase variable.

Keywords: Singular Dirichlet boundary value problem; Positive solution; Dead core solution; Pseudodead core solution; Existence

1. Introduction

Let A and T be positive numbers. We consider the singular Dirichlet boundary value problem (BVP)

$$(\phi(u'(t)))' = \lambda[f(t,u(t),u'(t)) + h(t,u(t),u'(t))], \tag{1.1$^\lambda$}$$

$$u(0) = A, \qquad u(T) = A \tag{1.2}$$

depending on the positive parameter λ. Here $\phi \in C^0(\mathbb{R})$ is increasing, $f \in C^0([0,T] \times (0,A] \times \mathbb{R})$, $h \in C^0([0,T] \times [0,A] \times (\mathbb{R} \setminus \{0\}))$, f is singular at the value 0 of its first phase variable and h may be singular at the value 0 of its second phase variable.

We say that $u \in C^1[0,T]$ is *a positive solution of the BVP* (1.1)$^\lambda$ and (1.2) if $\phi(u') \in C^1([0,T] \setminus \mathcal{A})$ where $\mathcal{A} = \{t \in [0,T] : u'(t) = 0\}$, u satisfies (1.2), $0 < u \le A$ on $[0,T]$ and (1.1)$^\lambda$ holds for each $t \in [0,T] \setminus \mathcal{A}$.

A function $u \in C^1[0,T]$ is said to be *a dead core solution of the BVP* (1.1)$^\lambda$ and (1.2) if there exist $0 < \alpha < \beta < T$ such that $u(t) = 0$ for $t \in [\alpha,\beta]$, $0 < u \le A$ for $t \in [0,\alpha) \cup (\beta,T]$, $\phi(u') \in C^1([0,T] \setminus [\alpha,\beta])$, u satisfies (1.2) and (1.1)$^\lambda$ holds for each $t \in [0,T] \setminus [\alpha,\beta]$. The interval $[\alpha,\beta]$ is called the *dead core* of u (see e.g. [1,2]).

If $\alpha = \beta$ in the definition of the dead core solution u of the BVP (1.1)$^\lambda$ and (1.2), then u is said to be *a pseudodead core solution of the BVP* (1.1)$^\lambda$ and (1.2) (see [3]).

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Let $\|x\| = \max\{|x(t)| : 0 \le t \le T\}$ denote the norm in $C^0[0,T]$, $L_1[0,T]$ be the set of Lebesgue integrable functions on $[0,T]$ and $AC[0,A]$ be the set of absolutely continuous functions on $[0,A]$.

The aim of this paper is to discuss the existence of positive, dead core and pseudodead core solutions of the BVP $(1.1)^\lambda$ and (1.2).

Throughout the paper, the functions ϕ, f and h satisfy the following assumptions

(H_1) $\phi \in C^0(\mathbb{R})$ is odd and increasing, $\lim_{x\to\infty}\phi(x) = \infty$,

$$(H_2)\ \begin{cases} f \in C^0([0,T]\times(0,A]\times\mathbb{R}) \text{ is positive, } \lim_{x\to 0^+} f(t,x,y) = \infty \\ \text{for each } (t,y) \in [0,T]\times\mathbb{R} \text{ and} \\ f(t,x,y) \le p(x)\omega(|y|) \text{ for } (t,x,y) \in [0,T]\times(0,A]\times\mathbb{R}, \text{ where} \\ p : (0,A] \to (0,\infty) \text{ is nonincreasing, } p \in L_1[0,A] \text{ and } \omega : [0,\infty) \to (0,\infty) \\ \text{is nondecreasing,} \end{cases}$$

$$(H_3)\ \begin{cases} h \in C^0([0,T]\times[0,A]\times(\mathbb{R}\setminus\{0\})) \text{ is nonnegative and} \\ h(t,x,y) \le K\gamma(|y|) \text{ for } (t,x,y) \in [0,T]\times[0,A]\times(\mathbb{R}\setminus\{0\}) \\ \text{where } K \in [0,\infty) \text{ and } \gamma : (0,\infty) \to (0,\infty) \text{ is nonincreasing,} \end{cases}$$

(H_4) $\int_0^\infty \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s))+\gamma(\phi^{-1}(s))}\mathrm{d}s = \infty$.

For each $n \in \mathbb{N}$, define the functions $f_n \in C^0([0,T]\times[0,A]\times\mathbb{R})$, $f_n^* \in C^0([0,T]\times\mathbb{R}^2)$, $h_n \in C^0([0,T]\times[0,A]\times\mathbb{R})$ and $h_n^* \in C^0([0,T]\times\mathbb{R}^2)$ by the formulas

$$f_n(t,x,y) = \begin{cases} \min\{f(t,x,y), nx\} & \text{for } (t,x,y) \in [0,T]\times(0,A]\times\mathbb{R} \\ 0 & \text{for } (t,y) \in [0,T]\times\mathbb{R} \text{ and } x = 0, \end{cases}$$

$$f_n^*(t,x,y) = \begin{cases} f_n(t,A,y) & \text{for } (t,x,y) \in [0,T]\times(A,\infty)\times\mathbb{R} \\ f_n(t,x,y) & \text{for } (t,x,y) \in [0,T]\times[0,A]\times\mathbb{R} \\ x & \text{for } (t,x,y) \in [0,T]\times(-\infty,0)\times\mathbb{R}, \end{cases}$$

$$h_n(t,x,y) = \begin{cases} \min\{h(t,x,y), n|y|\} & \text{for } (t,x,y) \in [0,T]\times[0,A]\times(\mathbb{R}\setminus\{0\}) \\ 0 & \text{for } (t,x) \in [0,T]\times[0,A] \text{ and } y = 0 \end{cases}$$

and

$$h_n^*(t,x,y) = \begin{cases} h_n(t,A,y) & \text{for } (t,x,y) \in [0,T]\times(A,\infty)\times\mathbb{R} \\ h_n(t,x,y) & \text{for } (t,x,y) \in [0,T]\times[0,A]\times\mathbb{R} \\ h_n(t,0,y) & \text{for } (t,x,y) \in [0,T]\times(-\infty,0)\times\mathbb{R}. \end{cases}$$

Then

$$f_n(t,x,y) \le f(t,x,y) \le p(x)\omega(|y|), \quad (t,x,y) \in [0,T]\times(0,A]\times\mathbb{R}, \tag{1.3}$$

$$h_n(t,x,y) \le h(t,x,y) \le K\gamma(|y|), \quad (t,x,y) \in [0,T]\times[0,A]\times(\mathbb{R}\setminus\{0\}). \tag{1.4}$$

Since $f_n^*(t,0,y) = 0$ for $(t,y) \in [0,T]\times\mathbb{R}$, $h_n^*(t,x,0) = 0$ for $(t,x) \in [0,T]\times\mathbb{R}$, $\lim_{n\to\infty} f_n^*(t,x,y) = f(t,x,y)$ uniformly on each compact set of $[0,T]\times(0,A]\times\mathbb{R}$ and $\lim_{n\to\infty} h_n^*(t,x,y) = h(t,x,y)$ uniformly on each compact set of $[0,T]\times[0,A]\times(\mathbb{R}\setminus\{0\})$, we discuss the existence of positive, dead core and pseudodead core solutions of the BVP $(1.1)^\lambda$ and (1.2) by considering solutions of the family of auxiliary regular differential equations

$$(\phi(u'(t)))' = \lambda[f_n^*(t,u(t),u'(t)) + h_n^*(t,u(t),u'(t))] \tag*{$(1.5)_n^\lambda$}$$

with $n \to \infty$. We note that this technique to obtain positive, dead core and pseudodead core solutions of the BVP $(1.1)^\lambda$ and (1.2) is related to the technique presented in [3] where the authors discuss positive and dead core solutions to the BVP

$$u''(t) + r(t,u(t)) = \lambda g(t,u(t)),$$
$$u'(a) = 0, \qquad \beta u'(b) + \alpha u(b) = A, \quad \beta \ge 0,\ \alpha, A > 0,$$

where $r \in C^0((a,b]\times[0,\infty))$ is nonnegative, $r(t,0) = 0$ for $t \in (a,b]$ and $g \in C^0([a,b]\times(0,\frac{A}{\alpha}])$ is positive (see [3, Theorem 17]).

By *a solution of the BVP* $(1.5)_n^\lambda$ and (1.2) we mean a function $u_n \in C^1[0,T]$ such that $\phi(u_n') \in C^1[0,T]$, u_n satisfies (1.2) and $(1.5)_n^\lambda$ (with $u = u_n$) holds for each $t \in [0,T]$.

It is useful to introduce also the notion of a solution of the BVP $(1.1)^\lambda$ and (1.2). We say that u is *a solution of the BVP* $(1.1)^\lambda$ and (1.2) if there exists a sequence $\{k_n\} \subset \mathbb{N}$, $\lim_{n\to\infty} k_n = \infty$ such that $\lim_{n\to\infty} u_{k_n} = u$ in $C^1[0,T]$ where u_{k_n} is a solution of the BVP $(1.5)_{k_n}^\lambda$, (1.2). In Section 3 (see Theorem 3.1) we will show that any solution of the BVP $(1.1)^\lambda$ and (1.2) is either a positive solution or a dead core solution or a pseudodead core solution of this problem.

For the solvability of the BVP $(1.5)_n^\lambda$ and (1.2), we use the following existence principle which is the special case of the existence principle presented in [4, Theorem 2.1].

Theorem 1.1. *Let $\lambda \in (0,\infty)$ and $n \in \mathbb{N}$. Suppose that there exist positive constants S_0 and S_1 independent of μ such that $\|u\| < S_0$ and $\|u'\| < S_1$ for any solution u to the one parameter differential equations*

$$(\phi(u'(t)))' = \mu\lambda[f_n^*(t,u(t),u'(t)) + h_n^*(t,u(t),u'(t))], \quad \mu \in [0,1] \tag{1.6}$$

satisfying (1.2). Then the BVP $(1.5)_n^\lambda$ and (1.2) has a solution.

The rest of the paper is organized as follows. Section 2 is devoted to the regular BVP $(1.5)_n^\lambda$ and (1.2). Using Lemmas 2.1 and 2.4 and Theorem 1.1, it is proved that this problem has a solution for each positive λ and $n \in \mathbb{N}$ (Lemma 2.5). Lemmas 2.6 and 2.7 give properties of solutions to the BVP $(1.5)_n^\lambda$ and (1.2) which are used in the next section. The main results are presented in Section 3. Under assumptions (H_1)–(H_4), for each positive λ, the BVP $(1.1)^\lambda$ and (1.2) has a solution. This solution is either a positive solution or a pseudodead core solution or a dead core solution (Theorem 3.1). For λ sufficiently small, the BVP $(1.1)^\lambda$ and (1.2) has only positive solutions (Corollary 3.2). If λ is sufficiently large, then the BVP $(1.1)^\lambda$ and (1.2) has only dead core solutions (Corollary 3.3). Two examples demonstrate the application of our existence results.

2. Auxiliary regular BVP $(1.5)_n^\lambda$ and (1.2)

Properties of solutions to the BVP $(1.5)_n^\lambda$ and (1.2) are given in the following lemma.

Lemma 2.1. *Let $\lambda > 0$ and let u_n be a solution of the BVP $(1.5)_n^\lambda$ and (1.2). Then*

$$0 \le u_n(t) \le A \quad \text{for } t \in [0,T], \tag{2.1}$$

there exist α_n, β_n, $0 < \alpha_n \le \beta_n < T$, such that

$$\left.\begin{array}{ll} u_n' < 0 \quad \text{for } t \in [0,\alpha_n), & u_n'(t) = 0 \quad \text{for } t \in [\alpha_n,\beta_n], \\ u_n'(t) > 0 \quad \text{for } t \in (\beta_n, T] & \end{array}\right\} \tag{2.2}$$

and u_n' is increasing on the intervals $[0,\alpha_n]$ and $[\beta_n,T]$. If $\alpha_n < \beta_n$ then

$$u_n(t) = 0 \quad \text{for } t \in [\alpha_n,\beta_n]. \tag{2.3}$$

Proof. Suppose that $\min\{u_n(t) : 0 \le t \le T\} = u_n(\tau) < 0$. Then $\tau \in (0,T)$, $u_n'(\tau) = 0$ and $(\phi(u_n'(t)))'_{t=\tau} \ge 0$, which contradicts $(\phi(u_n'(t)))'_{t=\tau} = \lambda f_n^*(\tau, x_n(\tau), 0) = \lambda u_n(\tau) < 0$. Hence $u_n \ge 0$ on $[0,T]$. Therefore $(\phi(u_n'(t)))' = \lambda[f_n^*(t,u_n(t),u_n'(t)) + h_n^*(t,u_n(t),u_n'(t))] \ge 0$ for $t \in [0,T]$ and since ϕ is increasing on $\mathbb{R}$ by (H_1), u_n' is nondecreasing on $[0,T]$. From $(\phi(u_n'(t)))'_{t=0,T} > 0$ and $u_n'(\xi_n) = 0$ for some $\xi_n \in (0,T)$ which follows from $u_n(0) = u_n(T) = A$, we have $u_n'(0) < 0$ and $u_n'(T) > 0$. Hence there exist $0 < \alpha_n \le \beta_n < T$ such that (2.2) is true and also $u_n < A$ on $(0,T)$. Since $0 < u_n \le A$ on $[0,\alpha_n) \cup (\beta_n,T]$, we have

$$(\phi(u_n'(t)))' = \lambda[f_n(t,u_n(t),u_n'(t)) + h_n(t,u_n(t),u_n'(t))] > 0$$

for $t \in [0,\alpha_n) \cup (\beta_n,T]$, and from $u_n'(\alpha_n) = 0 = u_n'(\beta_n)$ and (2.2) we see that $\phi(u_n')$ is increasing on the intervals $[0,\alpha_n]$ and $[\beta_n,T]$ and the same is true for u_n'.

Assume that $\alpha_n < \beta_n$. Then $u_n(t) = B$ for $t \in [\alpha_n,\beta_n]$ where $B \in [0,A)$. If $B > 0$ then

$$\begin{aligned} f_n(t,u_n(t),u_n'(t)) + h_n(t,u_n(t),u_n'(t)) &= f_n(t,B,0) + h_n(t,B,0) \\ &= \min\{f(t,B,0), nB\} \end{aligned}$$

for $t \in [\alpha_n, \beta_n]$ and therefore for these t we have

$$(\phi(u_n'(t)))' = \lambda \min\{f(t, B, 0), nB\} > 0,$$

which is impossible. Hence (2.3) holds. $\square$

Remark 2.2. Lemma 2.1 shows (see (2.1)) that any solution u_n of the BVP $(1.5)_n^\lambda$ and (1.2) satisfies

$$(\phi(u_n'(t)))' = \lambda[f_n(t, u_n(t), u_n'(t)) + h_n(t, u_n(t), u_n'(t))]$$

for $t \in [0, T]$.

Remark 2.3. From the proof of Lemma 2.1 it follows that if a solution u_n of the BVP $(1.5)_n^\lambda$ and (1.2) satisfies $u_n > 0$ on $[0, T]$ then u_n' has a unique zero on $[0, T]$.

We now give *a priori* bounds for solutions of the BVP $(1.5)_n^\lambda$ and (1.2).

Lemma 2.4. *There exists a positive constant S depending on $\lambda > 0$ and independent of n such that*

$$\|u_n'\| < S \tag{2.4}$$

for any solution u_n of the BVP $(1.5)_n^\lambda$ and (1.2).

Proof. Let u_n be a solution of the BVP $(1.5)_n^\lambda$ and (1.2). By Lemma 2.1, u_n satisfies (2.1) and (2.2) with some $0 < \alpha_n \le \beta_n < T$ and u_n' is nondecreasing on $[0, T]$. Hence

$$\|x_n'\| = \max\{|u_n'(0)|, u_n'(T)\}. \tag{2.5}$$

From (1.3), (1.4) and (2.2) and Remark 2.2, it follows that

$$(\phi(u_n'(t)))' \le \lambda[p(u_n(t))\omega(-u_n'(t)) + K\gamma(-u_n'(t))], \quad t \in [0, \alpha_n)$$

and

$$(\phi(u_n'(t)))' \le \lambda[p(u_n(t))\omega(u_n'(t)) + K\gamma(u_n'(t))], \quad t \in (\beta_n, T].$$

Now integrating

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega(-u_n'(t)) + \gamma(-u_n'(t))} > \lambda[p(u_n(t)) + K]u_n'(t) \tag{2.6}$$

from 0 to α_n and

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega(u_n'(t)) + \gamma(u_n'(t))} < \lambda[p(u_n(t)) + K]u_n'(t) \tag{2.7}$$

from β_n to T yields

$$\begin{aligned}\int_0^{\phi(|u_n'(0)|)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))} \mathrm{d}s &< \lambda \int_{u_n(\alpha_n)}^A (p(s) + K)\,\mathrm{d}s \\ &\le \lambda \int_0^A (p(s) + K)\,\mathrm{d}s \end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\int_0^{\phi(u_n'(T))} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))}\,\mathrm{d}s &< \lambda \int_{u_n(\beta_n)}^A (p(s) + K)\,\mathrm{d}s \\ &\le \lambda \int_0^A (p(s) + K)\,\mathrm{d}s. \end{aligned} \tag{2.9}$$

By (H_4), there exists a positive constant L (depending on λ) such that

$$\int_0^L \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))}\,\mathrm{d}s > \lambda \int_0^A (p(s) + K)\,\mathrm{d}s.$$

From this, (2.8) and (2.9) imply $\max\{\phi(|u_n'(0)|), \phi(u_n'(T))\} < L$. Consequently $\max\{|u_n'(0)|, u_n'(T)\} < \phi^{-1}(L)$ and using (2.5), (2.4) is true with $S = \phi^{-1}(L)$. $\square$

Lemmas 2.1 and 2.4 together with Theorem 1.1 yield the following existence result for the BVP $(1.5)_n^\lambda$ and (1.2).

Lemma 2.5. *Let $\lambda > 0$. Then for each $n \in \mathbb{N}$, there exists a solution u_n of the BVP $(1.5)_n^\lambda$ and (1.2) satisfying (2.1) and (2.2) with some $0 < \alpha_n \le \beta_n < T$.*

Proof. Fix $n \in \mathbb{N}$. The existence of a solution of the BVP $(1.5)_n^\lambda$ and (1.2) will be proved if, by Theorem 1.1, there exist positive constants S_0 and S_1 (depending on λ) such that $\|u\| < S_0$ and $\|u'\| < S_1$ for any solution u of the family of the BVPs (1.6) and (1.2) where $\mu \in [0, 1]$. If $\mu = 0$ then $u = A$ is the unique solution of the BVP (1.6) and (1.2). Let $\mu \in (0, 1]$ and S be a positive constant in Lemma 2.4. From the proofs of Lemmas 2.1 and 2.4, it follows that $0 \le u \le A$ on $[0, T]$ and $\|u'\| < S$ for any solution u of the BVP (1.6) and (1.2). Hence the assumptions of Theorem 1.1 are satisfied with $S_0 = A + 1$ and $S_1 = S$. Consequently there exists a solution u_n of the BVP $(1.5)_n^\lambda$ and (1.2) and, by Lemma 2.1, u_n satisfies (2.1) and (2.2). $\square$

Lemma 2.5 shows that for any $\lambda > 0$, the BVP $(1.5)_n^\lambda$ and (1.2) has a solution u_n for all $n \in \mathbb{N}$. The next three lemmas give some properties of solutions u_n which will be used in Section 3.

Lemma 2.6. *Let $\lambda > 0$ and let u_n be a solution of the BVP $(1.5)_n^\lambda$ and (1.2). Then $\{u_n'\}$ is equicontinuous on $[0, T]$.*

Proof. By Lemmas 2.1 and 2.4, there exists a positive constant S and there exist $0 < \alpha_n \le \beta_n < T$ such that for each $n \in \mathbb{N}$,

$$0 \le u_n(t) \le A, \quad t \in [0, T], \tag{2.10}$$

$$\|u_n'\| < S, \tag{2.11}$$

u_n' is nondecreasing on $[0, T]$, $u_n' < 0$ on $[0, \alpha_n)$, $u_n' = 0$ on $[\alpha_n, \beta_n]$ and $u_n' > 0$ on $(\beta_n, T]$. In addition, if $\alpha_n < \beta_n$ for some $n \in \mathbb{N}$, then $u_n(t) = 0$ for $t \in [\alpha_n, \beta_n]$.

In order to show that $\{u_n'\}$ is equicontinuous on $[0, T]$, define the functions $G \in C^0[0, \infty)$ and $P \in AC[0, A]$ by the formulas

$$G(v) = \int_0^{\phi(v)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))}\,\mathrm{d}s, \quad v \in [0, \infty), \tag{2.12}$$

$$P(v) = \int_0^v (p(s) + K)\,\mathrm{d}s, \quad v \in [0, A], \tag{2.13}$$

where $K > 0$ appears in (H_3). Let $0 \le t_1 < t_2 \le T$. If $t_2 \le \alpha_n$, integrating (see (2.6))

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega(-u_n'(t)) + \gamma(-u_n'(t))} > \lambda[p(u_n(t)) + K]u_n'(t), \quad t \in [0, \alpha_n) \tag{2.14}$$

over $[t_1, t_2]$ yields

$$0 < G(-u_n'(t_1)) - G(-u_n'(t_2)) < \lambda[P(u_n(t_1)) - P(u_n(t_2))]. \tag{2.15}$$

If $t_1 \ge \beta_n$, integrating (see (2.7))

$$\frac{(\phi(u_n'(t)))' u_n'(t)}{\omega(u_n'(t)) + \gamma(u_n'(t))} < \lambda[p(u_n(t)) + K]u_n'(t), \quad t \in (\beta_n, T] \tag{2.16}$$

over $[t_1, t_2]$ gives

$$0 < G(u_n'(t_2)) - G(u_n'(t_1)) < \lambda[P(u_n(t_2)) - P(u_n(t_1))]. \tag{2.17}$$

If $t_1 < \alpha_n < t_2$, integrating (2.14) from t_1 to α_n yields

$$0 < G(-u_n'(t_1)) < \lambda[P(u_n(t_1)) - P(u_n(\alpha_n))] \tag{2.18}$$

and if $t_1 < \beta_n < t_2$, integrating (2.16) from β_n to t_2 gives

$$0 < G(u_n'(t_2)) < \lambda[P(u_n(t_2)) - P(u_n(\beta_n))]. \tag{2.19}$$

Finally, if $\alpha_n < \beta_n$ and $\alpha_n \le t_1 < t_2 \le \beta_n$, we have $u_n'(t_1) = u_n'(t_2) = 0$. Since $\{u_n\}$ is equicontinuous on $[0, T]$ which follows from the boundedness of $\{u_n\}$ in $C^1[0, T]$ and P is increasing and absolutely continuous on $[0, A]$, we see that $\{P(u_n)\}$ is equicontinuous on $[0, A]$. From this, given $\varepsilon > 0$ we can find $\delta > 0$ such that for each $0 \le t_1 < t_2 \le T$, $t_2 - t_1 < \delta$, we have

$$|P(u_n(t_2)) - P(u_n(t_1))| < \varepsilon, \quad n \in \mathbb{N}. \tag{2.20}$$

We now show that $\{\hat{G}(u_n')\}$ is equicontinuous on $[0, T]$ where $\hat{G} \in C^0(\mathbb{R})$ is defined by the formula

$$\hat{G}(v) = \begin{cases} G(v) & \text{for } v \in [0, \infty) \\ -G(-v) & \text{for } v \in (-\infty, 0). \end{cases} \tag{2.21}$$

Let $0 \le t_1 < t_2 \le T$, $t_2 - t_1 < \delta$. Then from (2.15), (2.17) and (2.20) it follows that

$$0 < \hat{G}(u_n'(t_2)) - \hat{G}(u_n'(t_1)) < \lambda\varepsilon \quad \text{if } 0 \le t_1 < t_2 \le \alpha_n$$
$$0 < \hat{G}(u_n'(t_2)) - \hat{G}(u_n'(t_1)) < \lambda\varepsilon \quad \text{if } \beta_n \le t_1 < t_2 \le T.$$

If $\alpha_n = \beta_n$ then (see (2.18)–(2.20))

$$0 < \hat{G}(u_n'(t_2)) - \hat{G}(u_n'(t_1)) < 2\lambda\varepsilon \quad \text{if } t_1 < \alpha_n = \beta_n < t_2$$

and if $\alpha_n < \beta_n$ and either $t_1 < \alpha_n < t_2$ or $\alpha_n \le t_1 < t_2 \le \beta_n$ or $t_1 < \beta_n < t_2$, we have

$$0 \le \hat{G}(u_n'(t_2)) - \hat{G}(u_n'(t_1)) < 2\lambda\varepsilon.$$

Hence $0 \le \hat{G}(u_n'(t_2)) - \hat{G}(u_n'(t_1)) < 2\lambda\varepsilon$ for $t_1, t_2 \in [0, T]$, $0 < t_2 - t_1 < \delta$, and consequently $\{\hat{G}(u_n')\}$ is equicontinuous on $[0, T]$. Since $\hat{G}$ is continuous and increasing on $\mathbb{R}$ and $\|u_n'\| < S$ by (2.11), $\{u_n'\}$ is equicontinuous on $[0, T]$. □

Lemma 2.7. *There exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0]$*

$$\sup\{\|u_n'\| : n \in \mathbb{N}\} < \frac{A}{T} \tag{2.22}$$

where u_n is a solution of the BVP $(1.5)_n^\lambda$ *and* (1.2).

Proof. Let $\lambda_0 > 0$ satisfy

$$\lambda_0 < \left(\int_0^A (p(s) + K)\,\mathrm{d}s\right)^{-1} \int_0^{\phi(A/T)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))}\,\mathrm{d}s.$$

Then

$$\lambda_0 = \left(\int_0^A (p(s) + K)\,\mathrm{d}s\right)^{-1} \int_0^{\phi(A/T-\varepsilon)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))}\,\mathrm{d}s$$

where $\varepsilon \in (0, \frac{A}{T})$. Let $\lambda \in (0, \lambda_0]$ and u_n be a solution of the BVP $(1.5)_n^\lambda$ and (1.2). Since (see (2.5), (2.8) and (2.9)) $\|u_n'\| = \max\{|u_n'(0)|, u_n'(T)\}$ and

$$\int_0^{\phi(\|u_n'\|)} \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))}\,\mathrm{d}s < \lambda\left(\int_0^A (p(s)+K)\mathrm{d}s\right) \le \lambda_0\left(\int_0^A (p(s)+K)\mathrm{d}s\right),$$

we have $\|u_n'\| < \frac{A}{T} - \varepsilon$. Hence (2.22) is true. $\square$

Remark 2.8. Let $\lambda_0 > 0$ be given in Lemma 2.7. If $\lambda \in (0, \lambda_0]$ then $u_n > 0$ on $[0, T]$ for any solution u_n of the BVP $(1.5)_{n_0}^{\lambda_1}$ and (1.2). Suppose not. Then there exist $\lambda_1 \in (0, \lambda_0]$, $n_0 \in \mathbb{N}$ and a solution u_{n_0} of the BVP $(1.5)_n^\lambda$ and (1.2) such that $u_{n_0}(\xi) = 0$ for some $\xi \in (0, T)$. Hence $A = u_{n_0}(T) - u_{n_0}(\xi) = u_{n_0}'(\eta)(T-\xi)$ where $\eta \in (\xi, T)$. Therefore $u_{n_0}'(\eta) = \frac{A}{T-\xi} > \frac{A}{T}$, contrary to $\|u_{n_0}'\| < \frac{A}{T}$.

Lemma 2.9. *For each $c \in (0, T)$, there exists $\lambda_c > 0$ such that for all $\lambda > \lambda_c$,*

$$\lim_{n\to\infty} u_n(c) = 0 \tag{2.23}$$

where u_n is a solution of the BVP $(1.5)_n^\lambda$ and (1.2).

Proof. Fix $c \in (0, T)$. Let $\varepsilon \in (0, A)$ and set $\varrho = \min\{c, T-c\}$,

$$\Lambda = \inf\left\{f(t,x,y) : (t,x,y) \in [0,T] \times (0,A] \times \left[-\frac{2A}{\varrho}, \frac{2A}{\varrho}\right]\right\} > 0,$$
$$\lambda_c = \frac{2}{\Lambda\varrho}\phi\left(\frac{2A}{\varrho}\right).$$

Fix $\lambda \in (\lambda_c, \infty)$ and let u_n be a solution of the BVP $(1.5)_n^\lambda$ and (1.2). We claim that

$$u_n(c) < \varepsilon \quad \text{for } n > \frac{\Lambda}{\varepsilon}. \tag{2.24}$$

If (2.24) is not true, there exists $n_0 > \frac{\Lambda}{\varepsilon}$ such that $u_{n_0}(c) \ge \varepsilon$. The next part of the proof is divided into two cases.

Case 1. Suppose $u_{n_0}'(c) \le 0$. Since u_{n_0}' is nondecreasing on $[0, T]$ by Lemma 2.1, $u_{n_0}' \le 0$ on $[0, c]$. If $u_{n_0}'(\frac{c}{2}) < -\frac{2A}{c}$ then $u_{n_0}' < -\frac{2A}{c}$ on $[0, \frac{c}{2}]$ and consequently

$$u_{n_0}(0) = u_{n_0}\left(\frac{c}{2}\right) - \int_0^{c/2} u_{n_0}'(t)\,\mathrm{d}t > u_{n_0}\left(\frac{c}{2}\right) + A > A,$$

which is impossible. Hence

$$u_{n_0}'\left(\frac{c}{2}\right) \ge -\frac{2A}{c}, \qquad 0 \ge u_{n_0}'(t) \ge -\frac{2A}{c} \quad \text{for } t \in \left[\frac{c}{2}, c\right]. \tag{2.25}$$

Since $n_0 u_{n_0}(t) \ge n_0\varepsilon > \Lambda$ for $t \in [0, c]$, we have

$$f_{n_0}(t, u_{n_0}(t), u_{n_0}'(t)) \ge \Lambda \quad \text{for } t \in \left[\frac{c}{2}, c\right].$$

Then from

$$\begin{aligned}(\phi(u_{n_0}'(t)))' &= \lambda[f_{n_0}(t, u_{n_0}(t), u_{n_0}'(t)) + h_{n_0}(t, u_{n_0}(t), u_{n_0}'(t))]\\ &\ge \lambda f_{n_0}(t, u_{n_0}(t), u_{n_0}'(t)) \ge \lambda\Lambda\\ &> \lambda_c\Lambda\end{aligned} \tag{2.26}$$

for $t \in [\frac{c}{2}, c]$, it follows that $\phi(u_{n_0}'(c)) - \phi(u_{n_0}'(\frac{c}{2})) > \frac{\lambda_c\Lambda c}{2}$. From this

$$\phi\left(-u_{n_0}'\left(\frac{c}{2}\right)\right) > -\phi(u_{n_0}'(c)) + \frac{\lambda_c\Lambda c}{2} \ge \frac{\lambda_c\Lambda c}{2} = \frac{c}{\varrho}\phi\left(\frac{2A}{\varrho}\right) \ge \phi\left(\frac{2A}{c}\right).$$

Therefore $-u'_{n_0}(\frac{c}{2}) > \frac{2A}{c}$, contrary to (2.25).

Case 2. Suppose $u'_{n_0}(c) > 0$. Then u'_{n_0} is positive and increasing on $[c, T]$ by Lemma 2.1. If $u'_{n_0}(\frac{T+c}{2}) > \frac{2A}{T-c}$ then $u'_{n_0} > \frac{2A}{T-c}$ on $[\frac{T+c}{2}, T]$ and $u_{n_0}(T) = u_{n_0}(\frac{T+c}{2}) + \int_{(T+c)/2}^{T} u'_{n_0}(t)\,\mathrm{d}t > A$, which is impossible. Hence

$$0 < u'_{n_0}(t) \le \frac{2A}{T-c} \quad \text{for } t \in \left[c, \frac{T+c}{2}\right]. \tag{2.27}$$

Since $n_0 u_{n_0} \ge n_0\varepsilon > \Lambda$ on $[c, T]$, it follows that $f_{n_0}(t, u_{n_0}(t), u'_{n_0}(t)) \ge \Lambda$ for $t \in [c, \frac{T+c}{2}]$ and therefore (2.26) holds on $[c, \frac{T+c}{2}]$. Integrating $\phi(u'_{n_0}(t))' > \lambda_c \Lambda$ from c to $\frac{T+c}{2}$ yields

$$\phi\left(u'_{n_0}\left(\frac{T+c}{2}\right)\right) - \phi(u'_{n_0}(c)) > \frac{\lambda_c \Lambda (T-c)}{2}.$$

and therefore

$$\begin{aligned}\phi\left(u'_{n_0}\left(\frac{T+c}{2}\right)\right) &> \phi(u'_{n_0}(c)) + \frac{\lambda_c \Lambda (T-c)}{2} > \frac{\lambda_c \Lambda (T-c)}{2} \\ &= \frac{T-c}{\varrho}\phi\left(\frac{2A}{\varrho}\right) \ge \phi\left(\frac{2A}{T-c}\right).\end{aligned}$$

Hence $u'_{n_0}(\frac{T+c}{2}) > \frac{2A}{T-c}$, contrary to (2.27).

We have proved that for each $\varepsilon > 0$ and $n > \frac{\Lambda}{\varepsilon}$, $u_n(c) < \varepsilon$. Since $u_n(c) \ge 0$ for $n \in \mathbb{N}$, (2.23) is true. □

3. Main results and examples

Theorem 3.1. *Let assumptions* (H$_1$)–(H$_4$) *be satisfied. Then for every $\lambda > 0$, the BVP* $(1.1)^\lambda$ *and* (1.2) *has a solution. Moreover, any solution of the BVP* $(1.1)^\lambda$ *and* (1.2) *is either a positive solution or a pseudodead core solution or a dead core solution of this problem.*

Proof. Fix $\lambda > 0$. Let u_n be a solution of the BVP $(1.5)_n^\lambda$ and (1.2). The existence of u_n for each $n \in \mathbb{N}$ follows from Lemma 2.5. By Lemmas 2.1, 2.4 and 2.5, there exists a positive constant S and there exist $0 < \alpha_n \le \beta_n < T$ such that for each $n \in \mathbb{N}$,

$$\begin{aligned}&0 \le u_n(t) < A, \quad t \in (0, T),\\ &\|u'_n\| < S,\end{aligned} \tag{3.1}$$

u'_n is increasing on the intervals $[0, \alpha_n]$ and $[\alpha_n, T]$, $u'_n < 0$ on $[0, \alpha_n)$, $u'_n = 0$ on $[\alpha_n, \beta_n]$ and $u'_n > 0$ on $(\beta_n, T]$. In addition, if $\alpha_n < \beta_n$ then $u_n = 0$ on $[\alpha_n, \beta_n]$. Also $\{u'_n\}$ is equicontinuous on $[0, T]$ by Lemma 2.6. Going if necessary to a subsequence, we can assume, by the Arzelà–Ascoli theorem, that $\{u_n\}$ converges in $C^1[0, T]$ to some $u \in C^1[0, T]$. Hence u is a solution of the BVP $(1.1)^\lambda$ and (1.2).

Let now u be a solution of the BVP $(1.1)^\lambda$ and (1.2). Without loss of generality we can assume that $u = \lim_{n\to\infty} u_n$ in $C^1[0, T]$ where u_n is a solution of the BVP $(1.5)_n^\lambda$ and (1.2) and u_n has the properties presented in the opening part of the proof of this theorem. Then $u(0) = u(T) = A$ and u' is nondecreasing on $[0, T]$. The next part of the proof is divided into three cases.

Case 1. Suppose $\min\{u(t) : 0 \le t \le T\} > 0$. Then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\min\{u_n(t) : 0 \le t \le T\} \ge \varepsilon \quad \text{for } n \ge n_0. \tag{3.2}$$

Also (see Remark 2.3) $\alpha_n = \beta_n$ and α_n is the unique zero of u'_n for $n \ge n_0$. Put

$$\mu = \min\{f(t, x, y) : (t, x, y) \in [0, T] \times [\varepsilon, A] \times [-S, S]\}.$$

Then $\mu > 0$ by (H$_2$) and if $n_1 \in \mathbb{N}$ satisfies $n_1 \ge \max\{\frac{\mu}{\varepsilon}, n_0\}$ we have

$$(\phi(u'_n(t)))' \ge \lambda f_n(t, u_n(t), u'_n(t)) \ge \lambda\mu \tag{3.3}$$

for $t \in [0, T]$ and $n \geq n_1$. Hence $-\phi(u_n'(t)) = \phi(u_n'(\alpha_n)) - \phi(u_n'(t)) \geq \lambda\mu(\alpha_n - t)$ and therefore

$$u_n'(t) \leq -\phi^{-1}(\lambda\mu(\alpha_n - t)) \quad \text{for } t \in [0, \alpha_n] \text{ and } n \geq n_1. \tag{3.4}$$

An analogous reasoning shows that

$$u_n'(t) \geq \phi^{-1}(\lambda\mu(t - \alpha_n)) \quad \text{for } t \in [\alpha_n, T] \text{ and } n \geq n_1. \tag{3.5}$$

Passing if necessary to a subsequence we can assume that $\{\alpha_n\}$ is convergent and let $\lim_{n\to\infty} \alpha_n = \alpha$. Letting $n \to \infty$ in (3.4) and (3.5) yields

$$u'(t) \leq -\phi^{-1}(\lambda\mu(\alpha - t)), \quad t \in [0, \alpha], \tag{3.6}$$

$$u'(t) \geq \phi^{-1}(\lambda\mu(t - \alpha)), \quad t \in [\alpha, T]. \tag{3.7}$$

Hence α is the unique zero of u' and $u(0) = u(T) = A$ shows that $\alpha \in (0, T)$. Furthermore,

$$\lim_{n\to\infty} f_n(t, u_n(t), u_n'(t)) = f(t, u(t), u'(t)), \quad t \in [0, T],$$
$$\lim_{n\to\infty} h_n(t, u_n(t), u_n'(t)) = h(t, u(t), u'(t)), \quad t \in [0, T] \setminus \{\alpha\}$$

and (see (1.3), (3.1) and (3.2))

$$0 < f_n(t, u_n(t), u_n'(t)) \leq p(\varepsilon)\omega(S), \quad t \in [0, T] \text{ and } n \geq n_1.$$

From (1.4), (3.1), (3.6) and (3.7), it follows that for $a \in (0, \alpha)$, $b \in (\alpha, T)$ and sufficiently large n,

$$0 \leq h_n(t, u_n(t), u_n'(t)) \leq K\gamma\left(\frac{1}{2}\phi^{-1}(\lambda\mu(\alpha - a))\right), \quad t \in [0, a]$$

and

$$0 \leq h_n(t, u_n(t), u_n'(t)) \leq K\gamma\left(\frac{1}{2}\phi^{-1}(\lambda\mu(b - \alpha))\right), \quad t \in [b, T].$$

Thus taking the limit as $n \to \infty$ in

$$\phi(u_n'(t)) = \phi(u_n'(0)) + \lambda\int_0^t [f_n(s, u_n(s), u_n'(s)) + h_n(s, u_n(s), u_n'(s))]\,\mathrm{d}s \tag{3.8}$$

and

$$\phi(u_n'(t)) = \phi(u_n'(T)) - \lambda\int_t^T [f_n(s, u_n(s), u_n'(s)) + h_n(s, u_n(s), u_n'(s))]\,\mathrm{d}s,$$

we get, by the Lebesgue dominated convergence theorem and using the fact that $a \in (0, \alpha)$ and $b \in (\alpha, T)$ are arbitrary,

$$\phi(u'(t)) = \phi(u'(0)) + \lambda\int_0^t [f(s, u(s), u'(s)) + h(s, u(s), u'(s))]\,\mathrm{d}s, \quad t \in [0, \alpha),$$

and

$$\phi(u'(t)) = \phi(u'(T)) + \lambda\int_t^T [f(s, u(s), u'(s)) + h(s, u(s), u'(s))]\,\mathrm{d}s, \quad t \in (\alpha, T].$$

Hence $\phi(u') \in C^1([0, T] \setminus \{\alpha\})$ and u satisfies $(1.1)^\lambda$ for $t \in [0, T] \setminus \{\alpha\}$. We have proved that u is a positive solution of the BVP $(1.1)^\lambda$ and (1.2).

Case 2. Suppose $\min\{u(t) : 0 \leq t \leq T\} = 0$, $u(\alpha) = u(\beta) = 0$ for some $0 < \alpha < \beta < T$ and $u > 0$ on $[0, \alpha) \cup (\beta, T]$. Since u' is nondecreasing on $[0, T]$, $u' < 0$ on $[0, \alpha)$ and $u' > 0$ on $(\beta, T]$. Let $t_0 \in (0, \alpha)$ and set $t_1 = \frac{\alpha + t_0}{2}$. Then there exists $n_2 \in \mathbb{N}$ such that $u_n(t) \geq u_n(t_1) \geq \frac{u(t_1)}{2}$ and $u_n'(t) < 0$ for $t \in [0, t_1]$ and $n \geq n_2$. Put

$$\varrho = \min\left\{f(t, x, y) : (t, x, y) \in [0, T] \times \left[\frac{u(t_1)}{2}, A\right] \times [-S, S]\right\}.$$

Then $\varrho > 0$ by (H_2). Let $n_3 \in \mathbb{N}$, $n_3 \geq \max\{\frac{2\varrho}{u(t_1)}, n_2\}$. Then $nu_n(t) \geq \frac{nu(t_1)}{2} \geq \varrho$ for $t \in [0, t_1]$ and $n \geq n_3$. Therefore for these t and n, $f_n(t, u_n(t), u_n'(t)) \geq \varrho$ and $(\phi(u_n'(t)))' \geq \lambda\varrho$. It follows that

$$\phi(u_n'(t)) = \phi(u_n'(t_1)) + \int_{t_1}^{t} (\phi(u_n'(s)))' \, ds < -\lambda\varrho(t_1 - t) \quad \text{for } t \in [0, t_1] \text{ and } n \geq n_3.$$

Hence $u_n'(t) < -\phi^{-1}(\lambda\varrho(t_1 - t_0))$ for $t \in [0, t_0]$ and $n \geq n_3$. Letting $n \to \infty$ gives $u'(t) \leq -\phi^{-1}(\lambda\varrho(t_1 - t_0))$ for $t \in [0, t_0]$. Consequently

$$\lim_{n\to\infty} f_n(t, u_n(t), u_n'(t)) = f(t, u(t), u'(t))$$

and

$$\lim_{n\to\infty} h_n(t, u_n(t), u_n'(t)) = h(t, u(t), u'(t))$$

for $t \in [0, t_0]$. Next it follows from (1.3) and (1.4) that

$$\sup\{f_n(t, u_n(t), u_n'(t)) : t \in [0, t_0],\ n \geq n_3\} \leq p\left(\frac{u(t_1)}{2}\right)\omega(S)$$

and

$$\sup\{h_n(t, u_n(t), u_n'(t)) : t \in [0, t_0],\ n \geq n_3\} \leq K\gamma[\phi^{-1}(\lambda\varrho(t_1 - t_0))].$$

Now taking the limit as $n \to \infty$ in (3.8), we get

$$\phi(u'(t)) = \phi(u'(0)) + \lambda \int_0^t [f(s, u(s), u'(s)) + h(s, u(s), u'(s))] \, ds \tag{3.9}$$

for $t \in [0, t_0]$ by the Lebesgue dominated convergence theorem. Since $t_0 \in (0, \alpha)$ is arbitrary, the equality (3.9) holds for $t \in [0, \alpha)$. Therefore $\phi(u') \in C^1[0, \alpha)$ and u satisfies $(1.1)^\lambda$ on the interval $[0, \alpha)$.

Essentially the same reasoning we can apply on the interval $(\beta, T]$ to obtain that $\phi(u') \in C^1(\beta, T]$ and u is a solution of $(1.1)^\lambda$ on $(\beta, T]$. Summarizing, we have shown that u is a dead core solution of the BVP $(1.1)^\lambda$ and (1.2) and $u(t) = 0$ for $[\alpha, \beta]$.

Case 3. Suppose $\min\{u(t) : 0 \leq t \leq T\} = 0$ and $u(\xi) = 0$ for a unique $\xi \in (0, T)$. On the intervals $[0, \xi)$ and $(\xi, T]$, we can proceed analogously to *Case* 2 (where the intervals $[0, \alpha)$ and $(\beta, T]$ are considered) in order to prove that $\phi(u') \in C^1([0, T] \setminus \{\xi\})$ and u satisfies $(1.1)^\lambda$ on $[0, T] \setminus \{\xi\}$. Therefore u is a pseudodead core solution of the BVP $(1.1)^\lambda$ and (1.2). □

Corollary 3.2. *Let assumptions* (H_1)–(H_4) *be satisfied. Then there exists* $\lambda_0 > 0$ *such that the BVP* $(1.1)^\lambda$ *and* (1.2) *has for* $\lambda \in (0, \lambda_0]$ *only positive solutions.*

Proof. Let $\lambda_0 > 0$ be given in Lemma 2.7. Let $\lambda \in (0, \lambda_0]$ and set $V = \sup\{\|u_n'\| : n \in \mathbb{N}\}$ where u_n is a solution of the BVP $(1.5)_n^\lambda$ and (1.2). Then $V < \frac{A}{T}$ by Lemma 2.7 and therefore $0 < A - VT \leq u_n(t) \leq A$ for $t \in [0, T]$ and $n \in \mathbb{N}$. Let u be a solution of the BVP $(1.1)^\lambda$ and (1.2). Since $u = \lim_{n\to\infty} u_{k_n}$ in $C^1[0, T]$ where $\{k_n\}$ is a subsequence of $\{n\}$, we see that $0 < A - VT \leq u \leq A$ on $[0, T]$. Hence all solutions of the BVP $(1.1)^\lambda$ and (1.2) are positive solutions for each $\lambda \in (0, \lambda_0]$. □

Corollary 3.3. *Let assumptions* (H_1)–(H_4) *be satisfied. Then for sufficiently large value of* λ*, the BVP* $(1.1)^\lambda$ *and* (1.2) *has only dead core solutions. Moreover, to given* c_1, c_2, $0 < c_1 < c_2 < T$*, the BVP* $(1.1)^\lambda$ *and* (1.2) *has for sufficiently large value of* λ *dead core solutions* u *such that* $u(t) = 0$ *for* $t \in [c_1, c_2]$.

Proof. Let $0 < c_1 < c_2 < T$. By Lemma 2.9, there exists $\lambda_* > 0$ such that for all $\lambda > \lambda_*$,

$$\lim_{n\to\infty} u_n(c_j) = 0, \quad j = 1, 2, \tag{3.10}$$

where u_n is a solution of the BVP $(1.5)_n^\lambda$ and (1.2). Let $\lambda > \lambda_*$ and u be a solution of the BVP $(1.1)^\lambda$ and (1.2). Then there exists a subsequence $\{k_n\}$ of $\{n\}$ such that $u = \lim_{n\to\infty} u_{k_n}$ in $C^1[0, T]$. Since u' is nondecreasing on $[0, T]$ and (3.10) gives $u(c_1) = u(c_2) = 0$, we have $u(t) = 0$ for $t \in [c_1, c_2]$. Hence u is a dead core solution. We have proved that for all $\lambda > \lambda_*$, any solution of the BVP $(1.1)^\lambda$ and (1.2) is a dead core solution. □

Example 3.4. Consider the differential equation

$$(|u'|^{p-2}u')' = \lambda\left(\frac{1}{u^\alpha} + tu^\beta|u'|^\delta + \frac{e^u}{|u'|^\nu}\right), \quad \lambda > 0, \tag{3.11}$$

where $p > 1$, $\alpha \in (0,1)$, $\beta, \nu \in (0,\infty)$ and $\delta \in (0,p)$. The Eq. (3.11) is the special case of $(1.1)^\lambda$ with $\phi(u) = |u|^{p-2}u$ satisfying (H_1) and $f(t,x,y) = \frac{1}{x^\alpha} + tx^\beta|y|^\delta$, $h(t,x,y) = \frac{e^x}{|y|^\nu}$. From $\frac{1}{x^\alpha} + tx^\beta|y|^\delta \le (\frac{1}{x^\alpha} + Tx^\beta)(1+|y|^\delta)$ we see that (H_2) is satisfied with $p(x) = \frac{1}{x^\alpha} + TA^\beta$ and $\omega(z) = 1 + z^\delta$. Also (H_3) is satisfied with $K = e^A$ and $\gamma(z) = \frac{1}{z^\nu}$. Since $\delta \in (0,p)$, we have

$$\begin{aligned}\int_0^\infty \frac{\phi^{-1}(s)}{\omega(\phi^{-1}(s)) + \gamma(\phi^{-1}(s))}\,ds &> \int_1^\infty \frac{s^{\frac{1}{p-1}}}{1 + s^{\frac{\delta}{p-1}} + s^{-\frac{\nu}{p-1}}}\,ds \\ &= \int_1^\infty \frac{s^{\frac{1+\nu}{p-1}}}{1 + s^{\frac{\nu}{p-1}} + s^{\frac{\delta+\nu}{p-1}}}\,ds \\ &> \frac{1}{3}\int_1^\infty s^{\frac{1-\delta}{p-1}}\,ds = \infty.\end{aligned}$$

By Theorem 3.1, for each $A > 0$, the BVP (3.11) and (1.2) has a solution which is either a positive solution or a pseudodead core solution or a dead core solution. If λ is sufficiently small then all solutions of the BVP (3.11) and (1.2) are positive solutions by Corollary 3.2, and if λ is sufficiently large then all solutions are dead core solutions by Corollary 3.3.

Example 3.5. Consider the singular BVP

$$u'' = \frac{\lambda}{\sqrt{u}}, \quad \lambda > 0, \tag{3.12}$$

$$x(0) = 1, \qquad x(T) = 1. \tag{3.13}$$

Eq. (3.12) is the special case of $(1.1)^\lambda$ with $\phi(u) = u$, $f(t,u) = \frac{1}{\sqrt{u}}$ and $h = 0$ satisfying assumptions (H_1)–(H_4) (where $p(u) = \frac{1}{\sqrt{u}}$, $\omega = 1$, $K = 0$ and $\gamma = 1$). Hence Theorem 3.1 guarantees that any solution of the BVP (3.12) and (3.13) is either a positive solution or a pseudodead core solution or a dead core solution of this problem. Assume that u is a dead core solution or a pseudodead core solution of the BVP (3.12) and (3.13) with some $\lambda_0 > 0$ in (3.12). Then there exist $0 < t_1 \le t_2 < T$ such that $u(t_1) = u'(t_1) = u(t_2) = u'(t_2) = 0$, $u = 0$ on $[t_1, t_2]$, $u' < 0$ on $[0, t_1)$ and $u' > 0$ on $(t_2, T]$. Integrating $u''(t)u'(t) = \frac{\lambda_0 u'(t)}{\sqrt{u(t)}}$ gives

$$(u'(t))^2 = 4\lambda_0\sqrt{u(t)} + a \quad \text{for } t \in [0, t_1]$$

where a is a constant. From $u(t_1) = u'(t_1) = 0$ we obtain $a = 0$. Hence $u'(t) = -2\sqrt{\lambda_0}\sqrt[4]{u(t)}$ for $t \in [0, t_1]$. Integrating the last equality over $[0,t] \subset [0,t_1]$ and using $u(0) = 1$ yields

$$u(t) = \left(1 - \frac{3}{2}\sqrt{\lambda_0}t\right)^{\frac{4}{3}} \quad \text{for } t \in [0, t_1]. \tag{3.14}$$

An analogous reasoning shows that

$$u(t) = \left(1 - \frac{3}{2}\sqrt{\lambda_0}(T-t)\right)^{\frac{4}{3}} \quad \text{for } t \in [t_2, T]. \tag{3.15}$$

From (3.14) and (3.15) it follows that $t_1 = t_2$ if and only if $\lambda_0 = \frac{16}{9T^2}$ and for $\lambda_0 > \frac{16}{9T^2}$ we have $t_2 - t_1 = T - \frac{4}{3\sqrt{\lambda_0}} > 0$.

Summarizing, we have proved that the BVP (3.12) and (3.13) has a positive solution for $\lambda \in (0, \frac{16}{9T^2})$, a unique pseudodead core solution for $\lambda = \frac{16}{9T^2}$ and dead core solutions for $\lambda > \frac{16}{9T^2}$. The function $u(t) = (1 - \frac{2t}{T})^{\frac{4}{3}}$ is the unique pseudodead core solution of the BVP (3.12) and (3.13) (with $\lambda = \frac{16}{9T^2}$ in (3.12)) and

$$u(t) = \begin{cases} \left(1 + \dfrac{3}{2}\sqrt{\lambda}t\right)^{\frac{4}{3}} & \text{for } t \in \left[0, \dfrac{2}{3\sqrt{\lambda}}\right] \\ 0 & \text{for } t \in \left[\dfrac{2}{3\sqrt{\lambda}}, T - \dfrac{2}{3\sqrt{\lambda}}\right] \\ \left(1 + \dfrac{3}{2}\sqrt{\lambda}(T-t)\right)^{\frac{4}{3}} & \text{for } t \in \left[T - \dfrac{2}{3\sqrt{\lambda}}, T\right] \end{cases}$$

is the unique dead core solution of the BVP (3.12) and (3.13) for $\lambda > \frac{16}{9T^2}$ and the interval $[\frac{2}{3\sqrt{\lambda}}, T - \frac{2}{3\sqrt{\lambda}}]$ is its dead core interval.

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